Kappa: A Generalized Downside Risk-Adjusted Performance Measure

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The Sortino Ratio and the more recently developed Omega statistic are conceptually related "downside" risk-adjusted return measures, but appear distinct mathematically. We show that each of these measures is a special case of Kappa, a generalized risk-adjusted performance measure. A single parameter of Kappa determines whether the Sortino Ratio, Omega, or another risk-adjusted return measure is generated.

Using shape estimation functions for investment return distributions, we show that values for the first four moments of a return distribution are sufficient in many cases to enable a robust estimation of Kappa: it is not necessary to know the individual data points in the distribution. This further parameterization of the Kappa calculation enables efficient risk-adjusted return measurements and comparisons among a broad range of investment alternatives, even in the absence of detailed returns data. We examine return rankings of hedge fund indices under several variations of Kappa, and the extent to which these are affected by higher moments of the return distribution.

The Sortino Ratio and Omega

The Sortino Ratio and the more recently-specified Omega statistic, as defined by Shadwick and Keating [2002], can be used as alternatives to the Sharpe ratio in measuring risk-adjusted return. Unlike Sharpe, neither assumes a normal return distribution, and each focuses on the *likelihood* of not meeting some target return. In contrast Sharpe measures only the sign and magnitude of the average risk premium relative to the risk incurred in achieving it.

As specified, the Sortino Ratio and Omega appear distinct. Omega is defined as:

$$\Omega(\tau) = \frac{\int_{\tau}^{\infty} [1 - F(R)] dR}{\int_{-\infty}^{\tau} F(R) dR} \tag{1}$$

where

F(.) = the cumulative density function (cdf) for total returns on an investment
τ = threshold return

The Sortino Ratio is defined as:

$$S = \frac{\mu - \tau}{\sqrt{\int_{-\infty}^{\tau} (\tau - R)^2 dF(R)}} \tag{2}$$

where

μ = the expected periodic return = $\int_{-\infty}^{\infty} R dF(R)$

Despite their apparent distinctiveness, Omega and the Sortino Ratio each represent a single case of a more generalized risk-return measure, defined below, which we call Kappa (K). K_n generates Omega when n=1, the Sortino Ratio when n=2, or any of an infinite number of related risk-return measures when n takes on any positive value. Kappa is undefined where n<=0.

Definition of Kappa

Harlow [1991] defines the n^{th} lower partial moment function as

$$LPM_n(\tau) = \int_{-\infty}^{\tau} (\tau - R)^n dF(R) \tag{3}$$

Substituting equation (3) into equation (2) provides an alternative, fully equivalent definition of the Sortino ratio as

$$S = \frac{\mu - \tau}{\sqrt[2]{LPM_2(\tau)}} \quad (4).$$

Kappa is a generalization of this quantity, thus:

$$K_n(\tau) = \frac{\mu - \tau}{\sqrt[n]{LPM_n(\tau)}} \quad (5)$$

by which definition, the Sortino Ratio is $K_2(\tau)$, with τ being the investor's minimum acceptable or "threshold" periodic return.

Regarding Omega, in Appendix A, we show that

$$\Omega(\tau) = \frac{\mu - \tau}{LPM_1(\tau)} + 1 \quad (6a)$$

or, equivalently,

$$\Omega(\tau) = K_1(\tau) + 1 \quad (6b)$$

Hence the Omega statistic and the Sortino ratio have identical structures, being equal to K_1+1 and K_2 respectively. Despite the addition of a constant to K_1 in the Shadwick and Keating definition of Omega, Omega and K_1 are for all practical purposes identical. We refer to them interchangeably below[1].

K_n is defined for any value of n exceeding zero. Thus, in addition to K_1 and K_2, any number of K_n statistics may be applied in evaluating competing investment alternatives or in portfolio construction.

Discrete Calculation vs. Curve-Fitting

The lower partial moment required to calculate Kappa can be estimated from a sample of actual returns by treating the sample observations as points in a discrete return distribution. Let

T = sample size
R_t = the t^{th} return observation

The estimated nth lower partial moment is

$$LPM_n(\tau) = \frac{1}{T}\sum_{t=1}^{T} \max\left[\tau - R_t, 0\right]^n \quad (7)$$

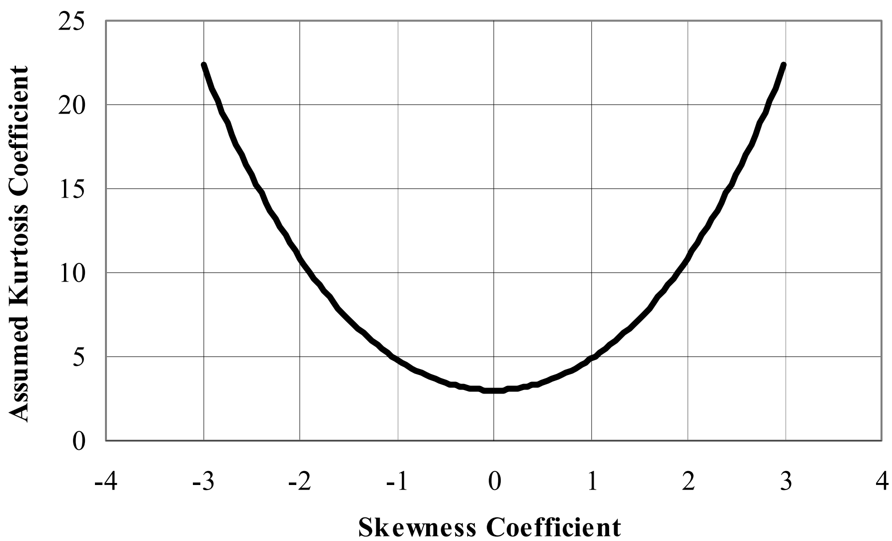
1 Kazemi, Schneeweis, and Gupta [2003] define a quantity that is similar to K_1 that they call the "Sharpe-Omega ratio."

This is a straightforward calculation but requires knowledge of the individual observations of the return sample. For obvious reasons relating to data availability, management and storage requirements, as well as for modeling and forecasting purposes, it would be useful to be able to calculate Kappa from a small number of return distribution characteristics rather than from raw data.

An alternative approach to estimating $LPM_n(\tau)$ is to assume that returns follow a particular continuous distribution and calculate the integral in equation (3) accordingly. For example, when estimating $LPM_2(\tau)$, Sortino [2001] and Forsey [2001] assume that returns follow the three-parameter lognormal distribution defined by Aitchinson and Brown [1957]. The three parameters of the distribution can be set so that the first three moments, mean (μ), standard deviation (σ), and coefficient of skewness (s) of the distribution match a given set of values.

For the three-parameter lognormal distribution, the fourth moment, the coefficient of kurtosis (κ), is a function of s. We show this relationship in Figure 1. However, the value of κ required to accurately represent the return distribution could in reality be higher or lower than the value implied by the three-parameter lognormal distribution. Johnson [1949] describes a three-member "family" of four-parameter distribution functions, including bounded and unbounded curves in addition to the three-parameter lognormal. Hill, Hill and Holder [1976] (hereafter HHH) extend the Johnson family of functions by adding the normal distribution, and provide an algorithm for selecting the appropriate function and estimating its parameters based on given values of the first four moments. We describe the extended Johnson family of distribution functions in Appendix B[2].

Figure 1: Skewness vs. Kurtosis, 3-Parameter Lognormal Distribution



Using the HHH algorithm, we create a continuous return distribution from the values of μ, σ, κ and s. Using the resulting distribution function, we estimate $LPM_n(\tau)$ by calculating the integral in equation (3) using the numerical techniques described by Press et al. [1992].

2 Kazemi, Schneeweis, and Gupta [2003] use an approximation of the Johnston distributions, but in a form that can result in the density function taking on negative values.

Johnson distribution functions have been used in other financial applications in a similar fashion. For example, Posner and Milevsky [1998] use Johnson functions to represent security price distributions and use them in integral formulas for pricing exotic options.

Testing Kappa Calculations Based On Fitted Curves

To evaluate how closely Kappa values derived from return distribution parameters match those derived from actual return data, K_1 and K_2 are calculated for 11 HFR monthly hedge fund indexes[3] for the period January 1990 through February 2003 ("the sample period"), using both actual return data and curves fitted as described above. Hedge fund indexes are used because they represent a broad spectrum of distribution shapes.

Distribution moments for the indexes are summarized in Table 1:

Table 1: Kappa Input Parameters of HFR Hedge Fund Indexes

Index	Average Monthly Excess Return	Standard Deviation	Coeff. of Skewness	Coeff. Of Kurtosis
HFR Convertible Arbitrage	0.550	0.994	-1.341	6.064
HFR Distressed Securities	0.756	1.835	-0.675	8.464
HFR Emerging Markets	0.763	4.533	-0.747	6.448
HFR Equity Hedge	1.026	2.678	0.132	4.172
HFR Equity Market Neutral	0.426	0.914	0.036	3.388
HFR Equity Non-hedge	0.861	4.276	-0.499	3.452
HFR Event-Driven	0.737	1.977	-1.387	7.761
HFR Fixed Income Arbitrage	0.310	1.335	-1.644	11.626
HFR Fund of Funds	0.434	1.700	-0.305	6.966
HFR Merger Arbitrage	0.491	1.284	-2.977	16.304
HFR Short Selling	0.131	6.523	0.042	4.124

K_1 and K_2 values for the indices, calculated using both discrete data and Johnson distribution functions estimated from the parameters above, appear in Tables 2a and 2b. Note that excess returns rather than raw returns are used; hence, a threshold value of $\tau = 0.0$ implies a target return equal to the risk-free rate.

Table 2a: Discrete & Parameter-Based Kappa Calculations, $\tau = 0.0\%$ / month

Index	K_1 Discrete	K_1 Parameter-Based	K_1 % Diff	K_2 Discrete	K_2 Parameter-Based	K_2 % Diff
HFR Convertible Arbitrage	2.81	2.70	3.67	0.91	0.94	2.63
HFR Distressed Securities	2.19	1.98	9.37	0.72	0.71	1.39
HFR Emerging Markets	0.56	0.56	0.36	0.25	0.25	1.20
HFR Equity Hedge	1.72	1.73	0.58	0.76	0.76	0.26
HFR Equity Market Neutral	2.45	2.29	6.46	1.00	0.98	1.11
HFR Equity Non-hedge	0.64	0.65	0.93	0.31	0.31	0.32
HFR Event-Driven	1.70	1.55	8.61	0.58	0.57	0.52
HFR Fixed Income Arbitrage	0.97	0.86	12.11	0.33	0.33	1.81
HFR Fund of Funds	1.03	0.99	3.41	0.43	0.42	2.34
HFR Merger Arbitrage	0.92	0.85	7.48	0.41	0.40	2.42

[3] Source: Hedge Fund Research, Inc.

HFR Short Selling	0.50	0.44	13.15	0.14	0.13	1.48

Table 2b: Discrete & Parameter-Based Kappa Calculations, □ = -1.0% / month

Index	K_1 Discrete	K_1 Parameter-Based	K_1 % Diff	K_2 Discrete	K_2 Parameter-Based	K_2 % Diff
HFR Convertible Arbitrage	21.255	24.824	16.79	4.632	4.811	3.86
HFR Distressed Securities	11.049	9.553	13.54	2.222	2.290	3.06
HFR Emerging Markets	1.733	1.728	0.29	0.667	0.657	1.50
HFR Equity Hedge	6.239	6.264	0.40	2.109	2.087	1.04
HFR Equity Market Neutral	48.036	58.155	21.07	9.183	10.146	10.49
HFR Equity Non-hedge	1.907	1.904	0.16	0.800	0.796	0.50
HFR Event-Driven	7.337	6.613	9.87	1.745	1.789	2.52
HFR Fixed Income Arbitrage	10.000	8.570	14.30	1.919	2.007	4.59
HFR Fund of Funds	8.505	7.645	10.11	2.091	2.064	1.29
HFR Merger Arbitrage	5.781	5.170	10.57	1.836	1.858	1.20
HFR Short Selling	8.419	7.257	13.80	1.509	1.552	2.85

For the hedge fund indices considered here, the percent differences between Kappa estimates based on discrete data and parameter-based Kappa estimates increase as τ decreases. These differences might be material if Kappa were used as an optimization metric, or for stress testing in a quantitative portfolio construction or asset allocation process. We conclude that parameter-based Kappa estimates should be interpreted with caution when used for these purposes.

However, possibly the commonest use of risk-adjusted performance measures is a simple comparison of competing investment alternatives. In this context, ranking differences that result from Kappa estimates based on the two calculation methods are a key consideration. Tables 3a through 3c show the rankings of the 11 hedge fund indices, for τ values of -1.0% per month, -0.5% per month and 0.0% per month respectively, for K_1, K_2 and K_3 over the sample period:

Table 3a: Index Rankings, Alternative Kappa Calculations, τ = -1.0% / month

	K_1		K_2		K_3	
	Discrete Data Rank	Parameter-Based Rank	Discrete Data Rank	Parameter-Based Rank	Discrete Data Rank	Parameter-Based Rank
HFR Convertible Arbitrage	10	10	10	10	10	10
HFR Distressed Securities	9	9	9	9	8	7
HFR Emerging Markets	1	1	1	1	1	1
HFR Equity Hedge	4	4	8	8	9	9
HFR Equity Market Neutral	11	11	11	11	11	11
HFR Equity Non-hedge	2	2	2	2	2	2
HFR Event-Driven	5	5	4	4	4	4
HFR Fixed Income Arbitrage	8	8	6	6	5	5
HFR Fund of Funds	7	7	7	7	6	6
HFR Merger Arbitrage	3	3	5	5	7	8
HFR Short Selling	6	6	3	3	3	3
Ranking Correlation	1.000		1.000		0.991	

Table 3b: Index Rankings, Alternative Kappa Calculations, τ= -0.5% / month

	K_1		K_2		K_3	
	Discrete Data Rank	Parameter-Based Rank	Discrete Data Rank	Parameter-Based Rank	Discrete Data Rank	Parameter-Based Rank
HFR Convertible Arbitrage	10	10	10	10	10	10
HFR Distressed Securities	9	9	9	9	8	8
HFR Emerging Markets	1	1	1	1	1	1
HFR Equity Hedge	4	6	8	8	9	9
HFR Equity Market Neutral	11	11	11	11	11	11
HFR Equity Non-hedge	2	2	2	2	2	2
HFR Event-Driven	7	8	6	7	6	5
HFR Fixed Income Arbitrage	8	7	5	5	4	4
HFR Fund of Funds	5	5	7	6	5	6
HFR Merger Arbitrage	3	3	4	4	7	7
HFR Short Selling	6	4	3	3	3	3
Ranking Correlation	0.955		0.991		0.991	

Table 3b: Index Rankings, Alternative Kappa Calculations, τ= 0.0% / month

	K_1		K_2		K_3	
	Discrete Data Rank	Parameter-Based Rank	Discrete Data Rank	Parameter-Based Rank	Discrete Data Rank	Parameter-Based Rank
HFR Convertible Arbitrage	11	11	10	10	10	10
HFR Distressed Securities	9	9	8	8	8	8
HFR Emerging Markets	2	2	2	2	2	2
HFR Equity Hedge	8	8	9	9	9	9
HFR Equity Market Neutral	10	10	11	11	11	11
HFR Equity Non-hedge	3	3	3	3	4	4
HFR Event-Driven	7	7	7	7	7	7
HFR Fixed Income Arbitrage	5	5	4	4	3	3
HFR Fund of Funds	6	6	6	6	5	5
HFR Merger Arbitrage	4	4	5	5	6	6
HFR Short Selling	1	1	1	1	1	1
Ranking Correlation	1.000		1.000		1.000	

Return rankings for the hedge fund indexes are, in general, little affected by the choice of Kappa estimation methodology. We conclude tentatively that, for the purpose of evaluating competing investment alternatives, the parameter-based method of estimating Kappa is robust and, where efficiency and simplicity are important, preferable to the more complex calculation based on discrete return data. However, more comprehensive tests of this conclusion are needed.

Return rankings *are*, however, affected materially by the choice of Kappa's n parameter. Note for instance that in Table 3b, the ranking of the Equity Hedge index ranges from 4 (K_1) to 9 (K_3). The rank of the Short Selling index is 6 under K_1 and 3 under K_3. These are the only two indices which combine positive skewness with a kurtosis coefficient of more than 4. In the next section we examine the relationship between Kappa values, skewness and kurtosis in more detail.

Table 4 summarizes the index rankings for K_1, K_2 and K_3 at a common return threshold. Note that, of the eleven indices examined, only two (Emerging Markets and Event-Driven) have constant ranks across all three Kappa variants. Consider also that these indices, too, might show ranking changes if other Kappa variants were used.

Table 4: Kappa Rankings of HFR Indices at $\tau = 0$

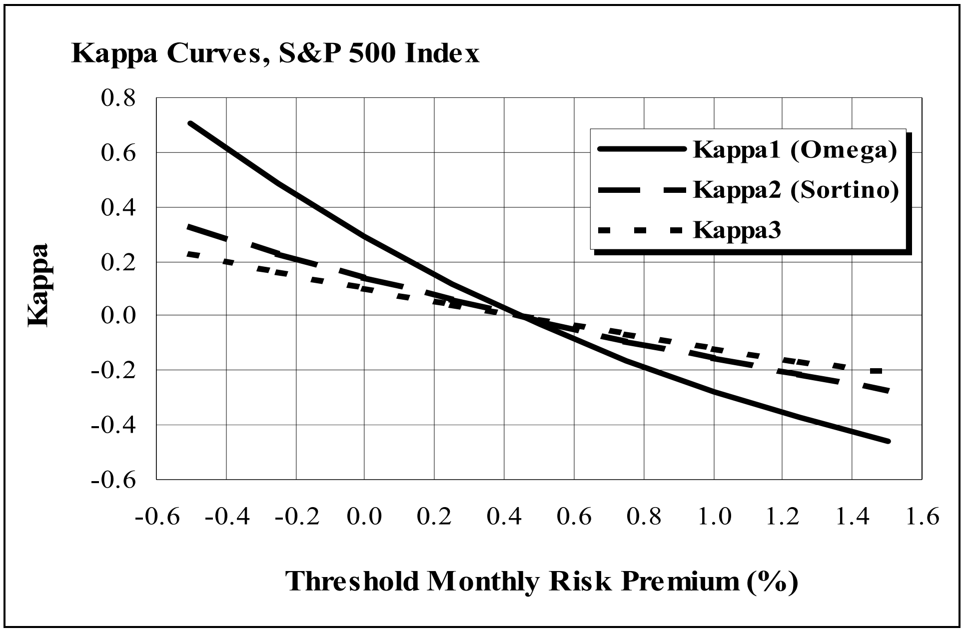
	K_1	K_2	K_3	Rank K_1	Rank K_2	Rank K_3
HFR Convertible Arbitrage	2.7026	0.9383	0.5781	1	2	2
HFR Distressed Securities	1.9829	0.7117	0.4204	3	4	4
HFR Emerging Markets	0.5610	0.2466	0.1606	10	10	10
HFR Equity Hedge	1.7250	0.7566	0.5118	4	3	3
HFR Equity Market Neutral	2.2889	0.9838	0.6714	2	1	1
HFR Equity Non-hedge	0.6497	0.3092	0.2181	9	9	8
HFR Event-Driven	1.5489	0.5721	0.3504	5	5	5
HFR Fixed Income Arbitrage	0.8562	0.3253	0.1951	7	8	9
HFR Fund of Funds	0.9906	0.4184	0.2660	6	6	7
HFR Merger Arbitrage	0.8542	0.4033	0.2881	8	7	6
HFR Short Selling	0.4358	0.1328	0.0769	11	12	12

The existence of substantive ranking differences implies that some Kappa variants may be superior to others in some circumstances. This has important implications regarding the choice of Kappa variant when evaluating investment alternatives or constructing a diversified portfolio. How the most appropriate Kappa variant should be chosen is, unfortunately, less obvious.

Empirical Behavior of Kappa

The Shape of Kappa Curves: All Kappa curves represent an inverse relationship between the threshold return chosen and the value of Kappa. The steepness of the K_n curve at any given threshold return is inversely related to the chosen value of the n parameter. Figure 2 shows K_1, K_2 and K_3 curves, relative to threshold return (expressed as a risk premium), for the S&P 500 Index, for the period January 1990 through February 2003.

Figure 2: Kappa Value vs. Threshold Risk Premium ($US)

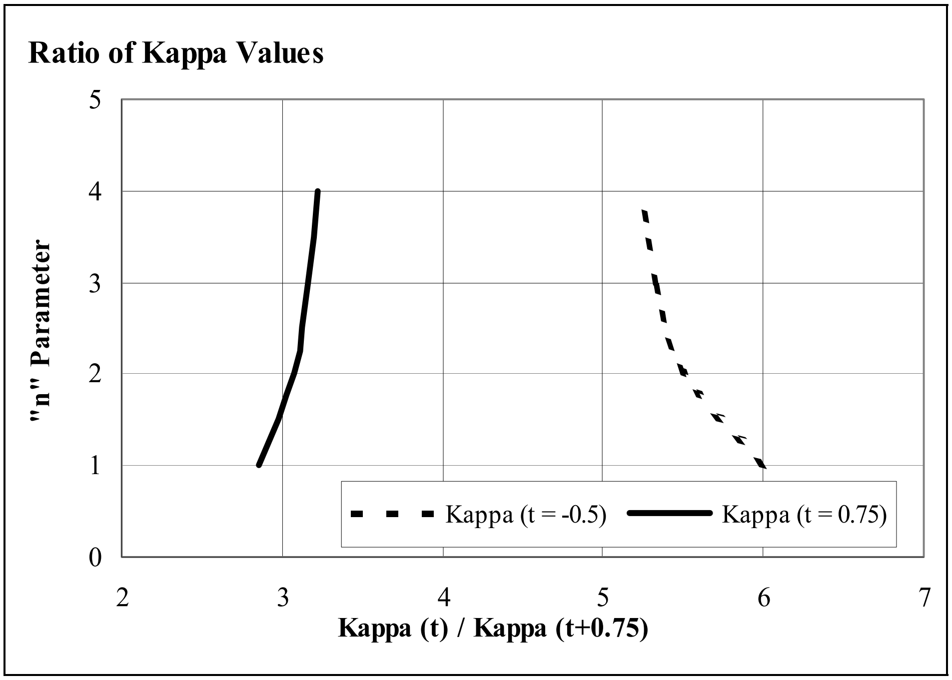


By construction, every Kappa variant has a value of zero when the threshold return equals the average figure.

Note that although K_2 – the Sortino ratio - is usually expressed as a single-point statistic relative to a single threshold return, it is more informative to plot this Kappa variant, as well as others, against a range of threshold values.

All Kappa curves are monotonic. Interpretation of differences in K_n values at different return thresholds is complex: although the value of K_1 at a threshold return of -0.4% is double that of K_1 at 0.0%, this does not make K_1 (-0.4) twice as "good" as K_1 (0.0). Differences between or ratios of Kappa values at different threshold returns are also dependent on the value of n chosen: the ratio of K_n (τ) to K_n $(\tau + 0.75)$, for the S&P 500 at different values of n, over the S&P sample period, is shown in the Figure 3. It can be seen that the rate of change in Kappa as a function of τ is inversely proportional to n.

Figure 3



We conclude tentatively that Kappa values should be treated as simple ordinals: for a given value of the n parameter, a higher Kappa is always better than a lower, but the size of the difference between two Kappa values is not subject to simple interpretation.

Kappa Sensitivity to Skewness: Kappa is insensitive to skewness for values of τ which lie close to or above the mean return; but sensitive to skewness when τ is substantially below the mean return. The charts below show the relationship between Kappa and skewness for a return distribution with a $\mu = 0.0$, $\sigma = 1.0$ and $\kappa = 3.0$. When $\tau = -2.0$, the ratio of K_1 at $s = 0.5$ to K_1 at $s = -0.5$ is about 46; the corresponding figures for K_2 and K_3 are 14 and 10 respectively. Thus, Kappa sensitivity to skewness, at low values of τ, is a negative function of n, and the n parameter can be interpreted as a measure of skewness risk "appetite" for threshold returns below the mean.

Figure 4a: Kappa Sensitivity to Skewness For n = 1, 2 and 3

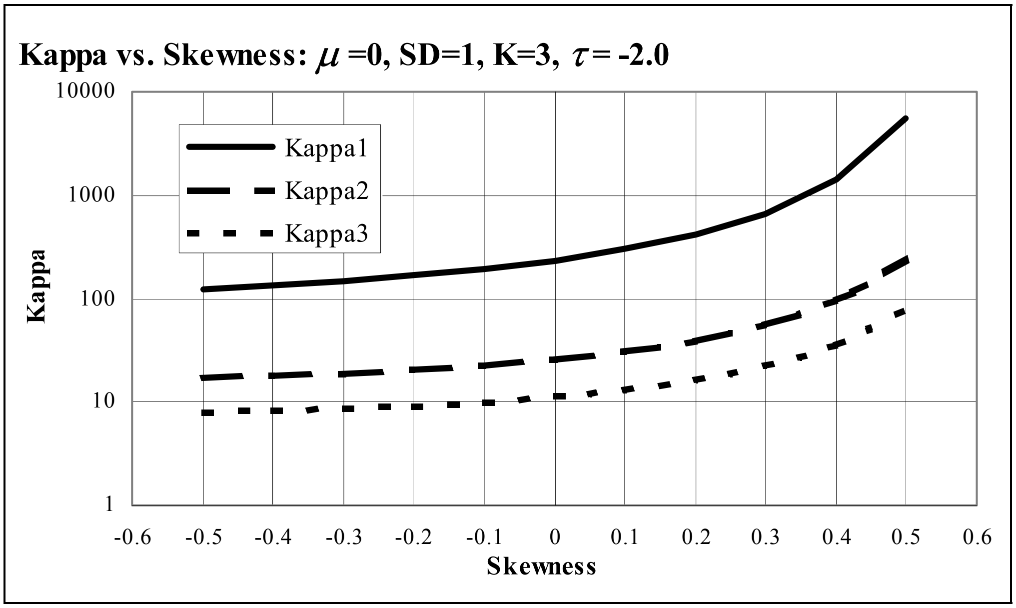
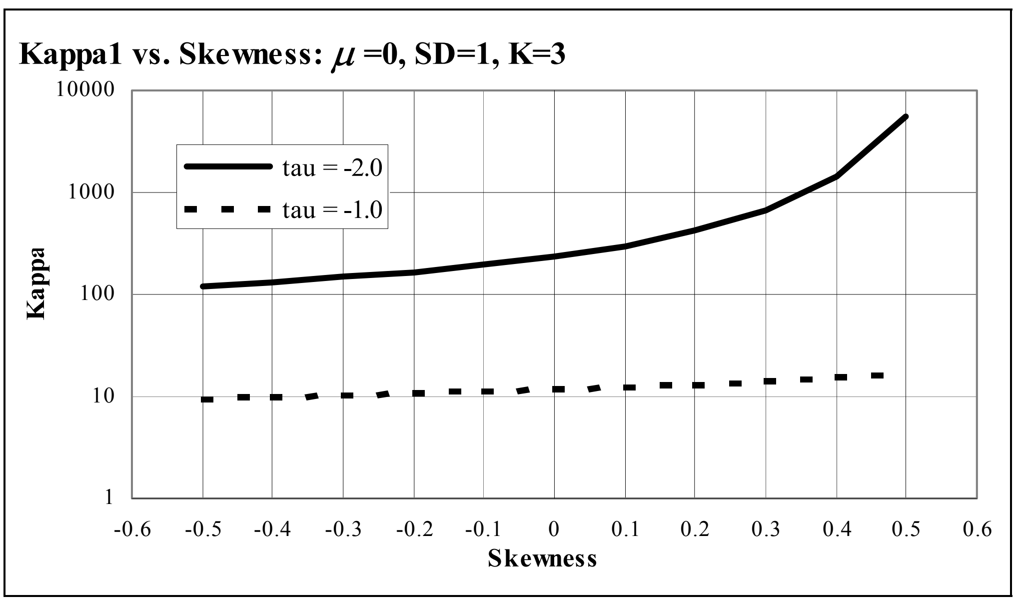


Figure 4b: K_1 Sensitivity to Skewness at Different Return Thresholds



Kappa Sensitivity to Kurtosis: As is the case with skewness, Kappa is insensitive to kurtosis for values of τ that lie close to or above the mean return; but sensitive to kurtosis when τ is substantially below the mean return. The charts below show the relationship between Kappa and kurtosis for a return distribution with $\mu = 0.0$, $\sigma = 1.0$ and $s = 0.0$. When $\tau = -2.0$, the ratio of K_1 at $\kappa = 5.0$ to K_1 at $\kappa = 3.0$ is about 0.53. The value of this ratio decreases slightly as n increases.

In this respect, it appears that the n parameter of Kappa can be interpreted as a measure of kurtosis risk aversion for return thresholds below the mean. This is in contrast to skewness, for which the n parameter appears to represent a measure of risk appetite.

Figure 5a: Kappa Sensitivity to Kurtosis For n = 1, 2 and 3

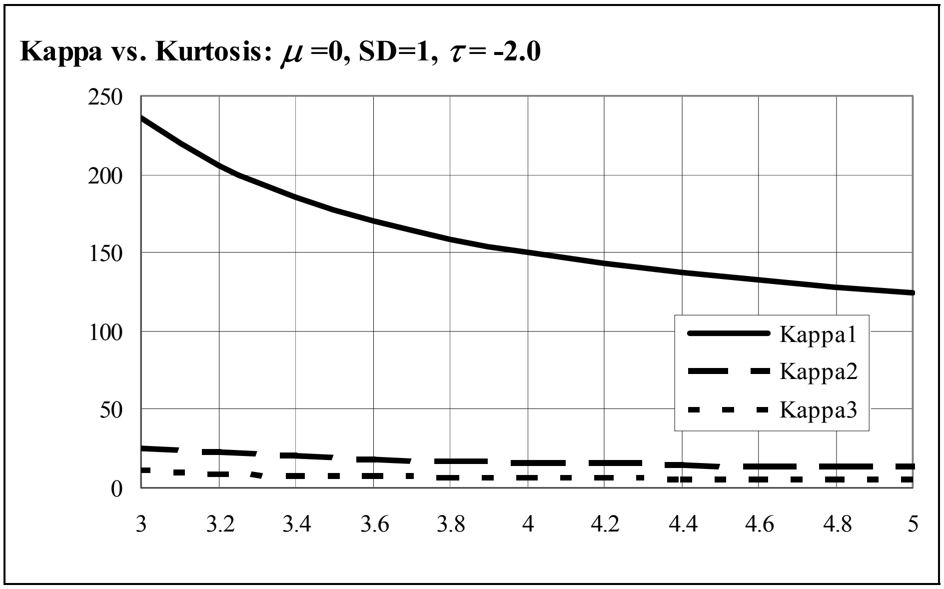
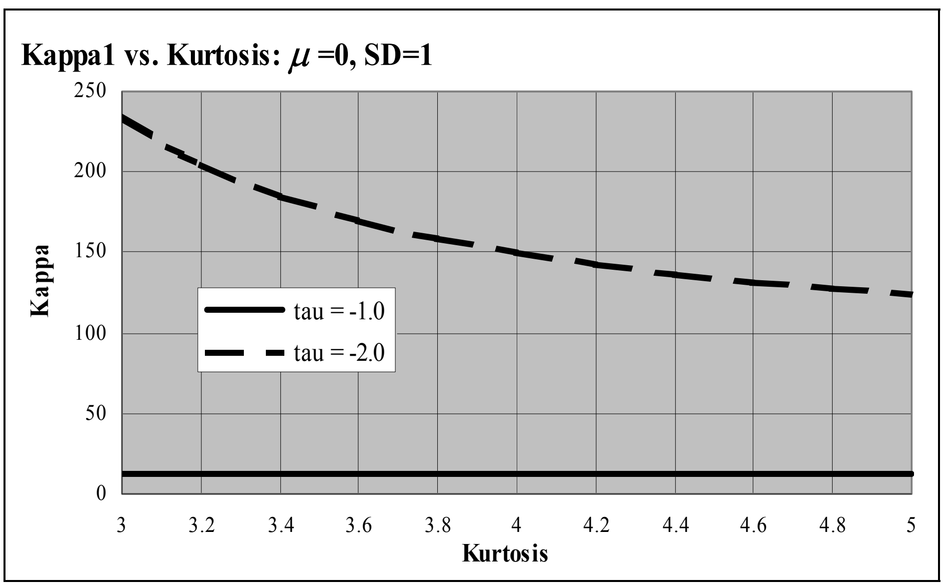


Figure 5b: K_1 Sensitivity to Kurtosis at Different Return Thresholds



Interpreting Kappa

Detailed derivations, descriptions and suggested applications of K_1 (Omega) and K_2 (Sortino ratio) already exist. There remains the question of what Kappa "means" in these cases or, as well, when n is set to some other value such as 0.5 or 3.

While interesting, the seeming relationship between Kappa variants and particular moments of a return distribution is not meaningful. For instance, while K_2 depends in part on semi-variance, and K_1 depends in part on a semi-mean, there is no corresponding distribution moment for Kappa variants with non-integer parameter values, nor is the relationship between K_3 and a notional "semi-skewness" statistic, or K_4 and a "semi-kurtosis" statistic, easily interpreted. Moreover, all Kappa variants are sensitive, to some degree, to the first four as well as other moments of the return distribution.

Conclusions

Omega and the Sortino ratio are two among many potential variants of Kappa. In certain circumstances, other Kappa variants may be more appropriate or provide more powerful insights.

The ranking of a given investment alternative can change according to the Kappa variant chosen, due in part to differences among the variants in their sensitivity to skewness and kurtosis. The choice of one Kappa variant over another will therefore materially affect the user's evaluation of competing investment alternatives, as well as the composition of any portfolio optimized to maximize the value of Kappa at some return threshold. We are not aware of any generally applicable rule for choosing the "correct" Kappa variant for a given purpose.

For the purposes of simple comparisons among competing investment alternatives at "ordinary" minimum return thresholds, Kappa may be estimated efficiently using a parameter-based calculation that eliminates the need to gather and manage discrete return data. However, this estimation method may lead to material discrepancies in Kappa estimates at low return thresholds, and so should be used with caution for the purposes of stress testing or portfolio construction. Other curve-fitting algorithms exist in addition to that described here, and some of these may potentially provide more robust parameter-driven Kappa estimates.

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Appendix A: Equivalence of Omega and Kappa_1

The Omega function is defined as

$$\Omega(\tau) = \frac{\int_{\tau}^{\infty} \left[1\text{-}F(R)\right] dR}{\int_{-\infty}^{\tau} F(R)\, dR} \tag{A.1}$$

Applying integration by parts to the denominator of $\Omega(\tau)$ yields:

$$\int_{-\infty}^{\tau} F(R)\, dR = \int_{-\infty}^{\tau} (\tau - R)\, dF(R) = LPM_1(\tau) \tag{A.2}$$

Applying integration by parts to the numerator of $\Omega(\tau)$ yields:

$$\int_{\tau}^{\infty} \left[1 - F(R)\right] dR = \int_{\tau}^{\infty} (R - \tau)\, dF(R) \tag{A.3}$$

From the definition of μ, it follows that

$$\mu - \tau = \int_{-\infty}^{\infty} (R - \tau)\, dF(R) = \int_{\tau}^{\infty} (R - \tau)\, dF(R) - \int_{-\infty}^{\tau} (\tau - R)\, dF(R) \tag{A.4}$$

Equation (A.3) shows that the second integral in equation (A.4) is the numerator of $\Omega(\tau)$. Equation (A.2) shows that the last integral in equation (A.4) is $LPM_1(\tau)$. Making these substitutions into equation (A.4) and rearranging terms yields:

$$\int_{\tau}^{\infty} \left[1 - F(R)\right] dR = \mu - \tau + LPM_1(\tau) \tag{A.5}$$

Substituting the right-hand side of equation (A.5) for the numerator of $\Omega(\tau)$ and $LPM_1(\tau)$ for the denominator of $LPM_1(\tau)$ yields:

$$\Omega(\tau) = \frac{\mu - \tau + LPM_1(\tau)}{LPM_1(\tau)} = \frac{\mu - \tau}{LPM_1(\tau)} + 1 \tag{A.6}$$

Hence,

$$\Omega(\tau) = K_1(\tau) + 1 \tag{A.7}$$

Appendix B: The Extended Johnson Family of Distribution Functions

A random variable X has a distribution function belonging to the Johnson family as defined by HHH if X can be transformed into a standard normal random variable Z as follows:

$$Z = g(X;\gamma,\delta,\xi,\lambda) \tag{B.1}$$

where g is one of the following four functions:

$$g(x;\gamma,\delta,\xi,\lambda) = \begin{cases} \lambda\left[\gamma + \delta \ln\left(\lambda(x-\xi)\right)\right], & \lambda x > \lambda\xi \; (\lambda = \pm 1) \quad \text{(3-param. lognormal)} \\ \gamma + \delta \sinh^{-1}\left(\dfrac{x-\xi}{\lambda}\right) & \text{(unbounded)} \\ \gamma + \delta \ln\left(\dfrac{x-\xi}{\xi+\lambda-x}\right), & \xi < x < \xi+\lambda \quad \text{(bounded)} \\ \gamma + \delta x & \text{(normal)} \end{cases} \tag{B.2}$$

The choice of g depends on the values of κ and *s*.[4] If $s \approx 0$ and $\kappa \approx 3$, we use the normal distribution and set

$$\delta = \frac{1}{\sigma} \tag{B.3}$$

$$\gamma = \frac{\mu}{\sigma} \tag{B.4}$$

Otherwise we use the procedure described below.

Let w be the solution to the equation

$$(w-1)(w+2)^2 = s^2 \tag{B.5}$$

Let

$$\kappa^* = w^4 + 2w^3 + 3w^2 - 3 \tag{B.6}$$

If X follows the three-parameter lognormal distribution, $\kappa = \kappa^*$. So if, $\kappa \approx \kappa^*$, we use the three-parameter lognormal distribution. The parameters are set as follows:

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4 Note that for κ and s to be a valid combination of coefficients of skewness and kurtosis, we must have $\kappa > s^2+1$.

$$\delta = \frac{1}{\sqrt{\ln(w)}} \tag{B.7}$$

$$\gamma = \frac{\delta}{2} \ln\left[\frac{w(w-1)}{\sigma^2}\right] \tag{B.8}$$

$$\lambda = \text{sign}(s) \tag{B.9}$$

$$\xi = \mu - \lambda \exp\left(\frac{\frac{1}{2\delta} - \gamma}{\delta}\right) \tag{B.10}$$

If κ is significantly less than κ*, the bounded distribution is used. If κ is significantly greater than κ*, the unbounded distribution is used. In these cases, the parameters are found using the iterative algorithms in HHH.